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The MONTHLY depends upon its present subscribers and loyal supporters to aid in spreading its influence and especially to assist in swelling its subscription list. It is important that new subscribers should begin with the January, 1913, issue, especially because this will contain the first instalment of Professor Cajori's latest research on "The History of the Logarithmic and Exponential Concepts," which is to be published serially during the coming months.

PARALLELOGRAMS INSCRIBED IN A RECTANGLE.

By L. A. HOWLAND, Wesleyan University, Middletown, Connecticut.

In the May, 1912, number of the MONTHLY, Professor Hodge points out the desirability of having a stock of a certain type of problems for use in a course in coördinate geometry. There is another type which seems to me to be also desirable for such a course. Not infrequently we find a student who has originality and is willing to think. His ability is not used to the full in the establishment of prescribed results. If started on the proper kind of problem, he is capable of discovering, what are to him at least, new results. The following seems to be such a problem, because its solution and

discussion suggest a large number of theorems, some of them by no means obvious or well known, yet discoverable by very elementary methods.

> *Problem:** Given a rectangle R of length $2a$ and breadth $2b$, to inscribe in it a parallelogram P, whose shorter side is h and whose angle with vertex on longer side of R is arc tan m.

We will take two sides of R as coördinate axes (Fig. 1), and let the coördinates of the vertices A and B be $(0, Y)$ and $(X, 0)$, respectively. Those of D and C will be $(2a-X, 2b)$ and $(2a, 2b-Y)$, respectively.
We have $X^2+Y^2=h^2$.

The equation of BC is $y=\frac{2b-Y}{2a-X}(x-X)$.

The equation of AB is $y=-\frac{Y}{X}(x-X)$.

If $\theta=\angle ABC=$arc tan m, we have

$$\tan\theta=-\frac{2bX-2XY+2aY}{2aX-X^2-2bY+Y^2}=m.$$

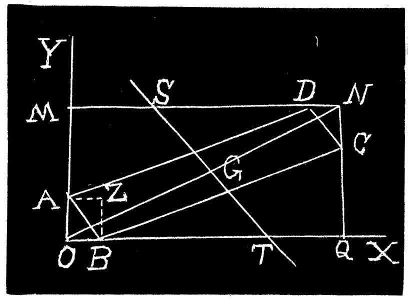

Fig. 1.

It appears then that X, Y must be solutions of the simultaneous equations

$$x^2+y^2=h^2\ldots(1),$$
$$mx^2+2xy-my^2-2(b+am)x-2(a-bm)y=0\ldots(2).$$

In other words, *the point Z is determined as the intersection of the circle* (1) *and the hyperbola* (2). From the position of Z we have at once the position of A and B and hence of C and D, and $ZN=l$, the length of P.

The hyperbola (2) is equilateral. Its center is the point (a, b), the center G of R. Its principal axis makes an angle $\frac{1}{2}$ arc tan$\frac{1}{m}$ with the X-axis. Its semi-axes are

$$\sqrt{\frac{m(a^2-b^2)+2ab}{\sqrt{(1+m^2)}}}.$$

Hence the vertices of these hyperbolas lie on the curve

$$r=\sqrt{\frac{m(a^2-b^2)+2ab}{\sqrt{(1+m^2)}}},$$

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* Suggested by the example in Osgood's *Calculus*, page 404.

$$\theta = \tfrac{1}{2} \text{ arc } \tan \frac{1}{m},$$

referred to G as pole and a parallel to the X-axis as initial line. Eliminating the parameter m, we have

$$r^2 = (a^2 - b^2) \cos 2\theta + 2ab \sin 2\theta.$$

If we take the diagonal of R as the initial line, we have $\theta = \phi - \text{arc } \tan \frac{b}{a}$, and the equation of the curve becomes

$$r^2 = (a^2 + b^2) \cos 2\phi.$$

This is a lemniscate with center at G and vertices at the vertices O and N of R.

The slope of the hyperbola at the origin is

$$s = \frac{b + am}{bm - a}.$$

Since the hyperbola lies on the opposite side of the tangent from its center, the branch of the curve through O (and, owing to symmetry, the other branch also) cannot cut into R, and hence there can be no P, unless $0 < s < \infty$. That is, unless

$$\begin{cases} b + am > 0 \\ bm - a > 0 \end{cases} \text{ or } \begin{cases} b + am < 0 \\ bm - a < 0 \end{cases}$$

This gives $m > \frac{a}{b}$ or $m < -\frac{b}{a}$.

These conditions may be expressed as follows: If $\alpha = \text{arc } \tan \frac{a}{b}$, then must $\angle ABC$ lie between α and $\alpha + 90°$. *The extent of the range of possible values for $\angle ABC$ is then always 90° entirely independent of the dimensions of R.*

For $m = -\frac{2ab}{a^2 - b^2}$, the hyperbola (2) becomes

$$(bx - ay)(ax + by - a^2 - b^2) = 0,$$

i. e., it degenerates into the diagonal ON and its perpendicular bisector ST. It can be shown that ST cuts none of the hyperbolas (2), — itself excepted, — in real points.

It is evident that when an h circle intersects the line ST, — and only then, — an inscribed rhombus will be determined. It is also evident that the maximum value of h is $OT = OS = \frac{a^2+b^2}{a}$.

A consideration of these results will suggest the following properties or theorems:

1. There can be no P inscribed in R unless $m > \frac{a}{b}$ or $m < -\frac{b}{a}$, *i. e.*, unless angle ABC lies between α and $\alpha + 90°$, where $\alpha = \text{arc tan}\frac{a}{b}$.

2. There can be no P inscribed in R unless $h \leqq \frac{a^2+b^2}{a}$.

3. For a given $h < \frac{a^2+b^2}{a}$ there will be an infinite number of P's, whose m's fill a definite interval, m_1 to m_2, dependent upon the size of h,

Problem: To determine this interval.

4. For a given m, satisfying the conditions of No. 1, there will be an infinite number of P's, whose h's vary from zero to a certain maximum value H, dependent upon the size of m.

Problem: To determine H.

5. For a given m, satisfying the conditions of No. 1, and a given h, less than or equal to the corresponding H, there will be one, and only one, P.

6. For a given m there may be two P's of the same length but of different h, or for a given h there may be two P's of the same length but of different m. In neither case can there be more than two.

Problem: To determine when two are possible.

7. In R there can be inscribed an infinite number of rhombuses, whose sides vary in length from $\sqrt{(a^2+b^2)}$ to $\frac{a^2+b^2}{a}$ inclusive. They all, however, have the same angles, $\angle ABC$ being $\arctan\left(-\frac{2ab}{a^2-b^2}\right)$.

8. No square can be inscribed in R unless R is itself a square.

9. In a square of side $2a$ there can be inscribed an infinite number of rectangles of breadth varying from 0 up to and including 2a. Of these, those whose breadth lies between $a\sqrt{2}$ and $2a$, both limits included, will be squares.

10. No rhombus can be inscribed in a square.

In the special case where a rectangle is to be inscribed in R, the hyperbola is

$$x^2 - y^2 - 2ax + 2by = 0.$$

Its axis is parallel to the X-axis and its semi-axis is $\sqrt{(a^2-b^2)}$. Since its eccentricity is $\sqrt{2}$, the distance of its foci from the center is $\sqrt{[2(a^2-b^2)]}$, and that of its directrices is $\frac{1}{2}\sqrt{[2(a^2-b^2)]}$. It may be neatly constructed as follows (Fig. 2):

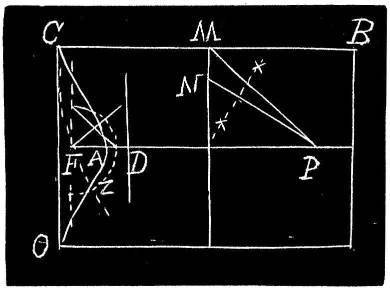

Fig. 2.

Lay off from M the distance $MP=a$. Then $O'P=\sqrt{(a^2-b^2)}$ is the distance from O' to the vertex. Lay off $O'N=O'P$. Then $NP=\sqrt{[2(a^2-b^2)]}$ is the distance from O' to the focus F, and $NQ=\frac{1}{2}NP$ is the distance from O' to the directrix. Draw a circle of any radius about F. Draw the chord of a quadrant and bisect it. The points of intersection of the circle and a line parallel to the directrix at a distance from it equal to the half chord lie on the hyperbola.

The applications of this problem are, of course, well known. For the sake of definiteness let us take the problem of finding the longest rectangle of a given width which can be laid on the floor of a rectangular room. If in Fig. 2 we draw a circle of radius $2a$ about B, it will cut the hyperbola at C and again at a point Z_1. If h is equal to or greater than OZ_1, BZ will obviously be equal to or less than $2a$. Hence if the rectangle has a width between OZ_1 and $2a$, it should be laid lengthwise of the room and can have a length $2a$. If its width is less than OZ_1, it should be "inscribed." Its length, BZ, will be greater than $2a$.

Fig. 2 is constructed for a rectangle 12×16 cm. It will obviously serve for any similar rectangle by proper choice of scale. OZ_1 measures 4.4+ cm.* Hence on the floor of a room $12'\times16'$ rectangles of width approximately $4'.4$ or wider can be but $16'$ long. Narrower rectangles can be longer. Measurement gives the length of one $3'$ wide to be $17.2'+$.*

NOTE ON LAMBERT'S METHOD OF SOLVING LINEAR DIFFERENTIAL EQUATIONS.

By H. L. SLOBIN, University of Minnesota.

In the July, 1910, *Annals*, Professor Lambert gives Cauchy's Method of Solving Linear Differential Equations, by breaking up the function

$$f(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \ldots,\ \frac{d^ny}{dx^n})=0 \ldots (1),$$

* Actual computation gives these values to four places of decimals to be: $OZ_1=4.4165$ and $BZ=17.2037$.